§4.2 Null Spaces, Column Spaces, and Linear Transformations

Recall that if A is an $m \times n$ matrix, the null space of A is the solution set to $Ax=0$.
In other words

$$\text{Nul } A = \{ x \mid x \text{ in } \mathbb{R}^n \text{ and } Ax = 0 \}$$

From §2.8 we know this is a subspace of $\mathbb{R}^n$ which we can verify quickly.

1) $A \cdot \underset{\substack{\text{zero}\\ \text{vector of } \mathbb{R}^n}}{0} = \underset{\substack{\text{zero vector}\\ \text{of } \mathbb{R}^m}}{0}$ so Nul$A$ contains zero vector (of $\mathbb{R}^n$)

2) If u, v are in Nul A, then $Au = 0$, $Av = 0$ so

$$A(u+v) = Au + Av = 0 + 0 = 0$$

so $u+v$ is in Nul A

3) If c is a scalar and u is in Nul A, then

$$A(cu) = c \cdot Au = c \cdot 0 = 0$$

so $c \cdot u$ is in Nul A

Recall we can find a basis for Nul A by row reduction. Hence we can write Nul A as a span of vectors of $\mathbb{R}^n$

Example

Describe Nul A if $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -2 & 1 & 4 \\ 0 & 1 & -2 & 1 & 2 \end{bmatrix}$

Solution: Want solution set for $Ax = 0$

$$A|0 = \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 1 & -2 & 1 & 4 & 0 \\ 0 & 1 & -2 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 1 & -2 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 7 & 2 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

x_3, x_4 free

$$\begin{cases} x_1 + 7x_3 + 2x_4 = 0 \\ x_2 - 2x_3 + x_4 = 0 \\ x_5 = 0 \end{cases} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -7 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{so } \operatorname{Nul} A = \operatorname{span}\left\{ \begin{bmatrix} -7 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

This is actually a basis for Nul A

Recall if A is an $m \times n$ matrix with columns $a_1, \dots, a_n$ the column space is

$$\operatorname{col} A = \operatorname{span}\{a_1, \dots, a_n\}$$

which is a subspace of $\mathbb{R}^m$

For example if $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -2 & 1 & 4 \\ 0 & 1 & -2 & 1 & 2 \end{bmatrix}$, then

$$\text{Col } A = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \right\}$$

$$= \text{span} \underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix} \right\}}_{\text{basis for Col } A}$$

since these are the pivot columns.

(see §2.8)

Example

Consider the subspace of $\mathbb{R}^3$ $W = \left\{ \begin{bmatrix} a + 2b \\ -3a \\ 4a - b \end{bmatrix} \middle| \ a, b \text{ real \#'s} \right\}$

Find a matrix such that Col $A = W$.

solution:

$$W = \left\{ a \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \middle| \ a, b \text{ in } \mathbb{R} \right\}$$

so $W = \text{col}\left(\begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 4 & -1 \end{bmatrix} \right)$

Defn: If V and W are vector spaces, a **linear transformation** is a map $T: V \longrightarrow W$ such that

1) $T(u+v) = T(u) + T(v)$ for all u, v in V

2) $T(c \cdot u) = c \cdot T(u)$ for all u in V and scalar c

Notice $T(0_V) = 0_W$ from this

- The **kernel** (or **null space**) of T is the set of all vectors u in V such that $T(u) = 0$

- The **range** of T is the set of all vectors in W of the form $T(v)$ for some v in V.

If $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a matrix transformation, i.e there's an $m \times n$ matrix with $T(x) = Ax$

- Kernel of T = null space of A, $\text{Nul}\, A$

- Range of T = column space of A, $\text{Col}\, A$

Example

Check this is a linear transformation!

Let $T: \mathbb{P}_2 \longrightarrow \mathbb{R}^2$ be the transformation given by

$$T(p(t)) = \begin{bmatrix} p(1) \\ p(0) \end{bmatrix}$$ What are the kernel and range of T?

poly of deg ≤ 2

$$\ker(T) = \left\{ p(t) \;\middle|\; \begin{bmatrix} p(1) \\ p(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \{ p(t) \mid p(1) = 0 \text{ and } p(0) = 0 \}$$

$$= \{ p(t) \mid t(t-1) \text{ divides } p(t) \}$$

$$= \text{span}\{ t(t-1) \}$$ since $\deg p(t) \leq 2$

$$\text{Range of } T = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \;\middle|\; a = p(1),\ b = p(0) \text{ for } p(t) \text{ in } \mathbb{P}_2 \right\}$$

$$= \mathbb{R}^2 \quad \text{why?}$$

Let $p(t) = b + (a-b)t$, then $T(p(t)) = \begin{bmatrix} a \\ b \end{bmatrix}$ where

a, b could be any real #s